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Means on Commutative Semigroups and Nonlinear Ergodic Theorems

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1. INTRODUCTION

Let C be a nonempty closed convex subset of a real Banach space E. Then, $T: C \to C$ is called nonexpansive on C, if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of a mapping T on C. Let $\mathscr{S} = \{S(t): t \geqslant 0\}$ be a family of nonexpansive mappings of C into itself such that $S(0) = I$, $S(t+s) = S(t)\, S(s)$ for all $t, s \in [0, \infty)$, and $S(t)\, x$ is continuous in $t \in [0, \infty)$ for each $x \in C$. Then, $\mathscr{S}$ is said to be a nonexpansive semigroup on C.

Baillon [1] proved the first nonlinear ergodic theorem for nonexpansive mappings in the framework of Hilbert spaces. This theorem was extended to Banach spaces by Baillon [2], Bruck [5], Hirano [7], and Reich [13]. Especially, Bruck [5] studied the asymptotic behavior of orbits of T independent of initial values and gave a simple proof of the ergodic theorem in Banach spaces. Bruck's proof is elegant and introduces a number of highly original ideas which are certain to find further applications. On the other hand, nonlinear ergodic theorems for a semigroup of nonexpansive mappings in a Hilbert space were studied by Brézis and Browder [3], Rodé [14], Takahashi [15], and the others. Especially Rodé found a sequence of means on the semigroup, generalizing the Cesàro means on positive integers, such that the corresponding sequence of mappings converges to a projection onto the set of common fixed points. Recently, Hirano, Kido, and Takahashi [9] established a nonlinear ergodic theorem of Rodé's type for semigroups of nonexpansive mappings in Banach spaces.

In this paper, by using ideas of Bruck [5], we study the asymptotic behavior of orbits of a semigroup of nonexpansive mappings in a Banach space. We first find a sequence $\{\mu_\alpha\}$ of means on the semigroup, generalizing the Cesàro means, such that, for each weak neighborhood W of the set of common fixed points, the corresponding mean vectors x_α are contained in W for sufficiently large α; see Theorem 1. This is a generalization of

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Bruck's theorem [5, Theorem 1]. Further, using this result, we obtain a nonlinear ergodic theorem which generalizes simultaneously the ergodic theorem for a single mapping and the corresponding theorem for a non-expansive semigroup in a Banach space. We can also use the result to study the asymptotic behavior of a solution of a nonlinear evolution equation; see Kido [10].

2. Preliminaries

Throughout this paper, we assume that a Banach space E is real. We denote by E^* the dual space of E and by R the set of all numbers. Let G be a semitopological semigroup with identity, i.e., G is a semigroup with the identity and Hausdorff topology such that the semigroup operator $G \times G \to G$ by $(s, t) \to s+t$ $(s, t \in G)$ is separately continuous. If G is commutative, $(G, \leqslant)$ is a directed system when the binary relation "$\leqslant$" on G is directed by $s \geqslant t$ if and only if $s = t+u$ for some $u \in G$. Let $cm(G)$ be the Banach space of all bounded continuous real valued functions on G with the supremum norm. We also denote by $m(G)$ the Banach space of all bounded real valued functions on G with the supremum norm. For each $s \in G$ and $f \in m(G)$, we define an element $r_s f$ in $m(G)$ by $r_s f(t) = f(t+s)$ for all $t \in G$. An element $\mu \in m(G)^*$ is called a mean on G if $\|\mu\| = \mu(1) = 1$. For every $f \in m(G)$ and $\mu \in m(G)^*$, we denote the value of μ at f by $\mu(f)$ or $\int f(s)\, d\mu(s)$ to specify the variable s. A mean μ on G is called invariant (resp. c-invariant) if $\mu(r_s f) = \mu(f)$ for all $s \in G$ and $f \in m(G)$ (resp. $f \in cm(G)$).

Let C be a closed convex subset of a reflexive Banach space E and G a commutative semitopological semigroup with the identity. Then, $\mathscr{G} = \{T_s : s \in G\}$ is called a continuous representation of G as nonexpansive mappings of C into itself if the following conditions are satisfied:

(a) $T_{s+t}(x) = T_s T_t(x)$ for all $s, t \in G$ and $x \in C$;

(b) the mapping $s \to T_s x$ is continuous for each $x \in C$.

We denote by $F(T_t)$ the set of fixed points of a mapping T_t and by $F(\mathscr{G})$ the set of common fixed points of mappings T_s.

Let μ be a mean on G and $\{\varphi_s : s \in G\}$ a bounded set of C. Then, we can define a continuous linear functional F on E^* by

$$F(x^*) = \int \langle \varphi_s, x^* \rangle \, d\mu(s)$$

for each $x^* \in E^*$, where $\langle x, x^* \rangle$ is the value of $x^* \in E^*$ at $x \in E$. Since E is reflexive, F is expressed by an element in the closure of the convex hull of

$\{\varphi_s: s \in G\}$, denoted by $\mu_s\langle\varphi_s\rangle$ or $\int \varphi_s \, d\mu(s)$. Especially, if $\varphi_s = T_s x$ for some $x \in C$, then $\int T_s x \, d\mu(s)$ is also denoted by $\mathcal{T}_\mu x$ for simplicity. If $\{T_s x: s \in G\}$ is bounded for every $x \in C$, then it is easy to see that $\mathcal{T}_\mu$ is nonexpansive on C. According to [9], a mean μ on G is said to be compact (resp. finite) if there exists a compact (resp. finite) subset S of G such that $\mu(1_S) = 1$, where $1_S(t) = 1$ on S and $1_S(t) = 0$ elsewhere. If a mean μ on G is finite, then it is expressed by $\sum_{i=1}^n a_i \delta_{s_i}$ for some $s_i \in G$ and $a_i \geqslant 0$ with $\sum_{i=1}^n a_i = 1$, where δ_t is a mean on G defined by $\delta_t(f) = f(t)$ for all $f \in m(G)$. Simultaneously, $\int \varphi_s \, d\mu(s)$ (resp. $\mathcal{T}_\mu x$) is expressed by $\sum_{i=1}^n a_i \varphi_{s_i}$ (resp. $\sum_{i=1}^n a_i T_{s_i} x$). Here, we give a proposition which is crucial for our discussion.

PROPOSITION. *Let $s \mapsto \varphi_s$ be a bounded continuous function from G into E with weak topology. Then, for any means μ, μ' on G,*

$$\left\| \int \varphi_s \, d\mu(s) - \int \varphi_s \, d\mu'(s) \right\| \leqslant \sup_{s \in G} \|\varphi_s\| \cdot \|\mu - \mu'\|_c, \tag{2.1}$$

where $\|\cdot\|_c$ is the norm of $cm(G)^*$.

Proof.

$$\begin{aligned}
&\left\| \int \varphi_s \, d\mu(s) - \int \varphi_s \, d\mu'(s) \right\| \\
&\leqslant \sup_{\|y^*\|=1} \left| \left\langle \int \varphi_s \, d\mu(s) - \int \varphi_s \, d\mu'(s), y^* \right\rangle \right| \\
&= \sup_{\|y^*\|=1} \left| \int \langle \varphi_s, y^* \rangle \, d\mu(s) - \int \langle \varphi_s, y^* \rangle \, d\mu'(s) \right| \\
&= \sup_{\|y^*\|=1} \left| \int \langle \varphi_s, y^* \rangle \, d(\mu - \mu')(s) \right| \\
&\leqslant \sup_{\|y^*\|=1} \sup_{s \in G} |\langle \varphi_s, y^* \rangle| \cdot \|\mu - \mu'\|_c \\
&\leqslant \sup_{s \in G} \|\varphi_s\| \cdot \|\mu - \mu'\|_c.
\end{aligned}$$

Let E be a Banach space. Then, with each $x \in E$, we associate the set

$$J(x) = \{x^* \in E^*: \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

Using the Hahn–Banach theorem it is immediately clear that $J(x) \neq \varnothing$ for each $x \in E$. Then the multivalued operator $J: E \to E^*$ is called the duality mapping of E. Let $B = \{x \in E: \|x\| = 1\}$. Then, the norm of E is said to be

Gâteaux differentiable (and E is said to be smooth) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for each x and y in B. It is said to be Fréchet differentiable if for each x in B, the limit (2.2) is attained uniformly for y in B. It is well known that if E is smooth, then the duality mapping J is single valued. And also we know that if E has a Fréchet differentiable norm, then J is norm to norm continuous.

For a subset D of E, $\overline{\text{co}}\, D$ denotes the closure of the convex hull of D.

3. Definitions and Lemmas

Unless otherwise specified, C denotes a bounded closed convex subset of a uniformly convex Banach space E and G denotes a commutative semitopological semigroup with the identity. Let $\mathscr{G} = \{T_s : s \in G\}$ be a continuous representation of G as nonexpansive mappings from C into C. Then, by [4], it is well known that $F(\mathscr{G})$ is nonempty.

Definition 1. G is said to have a generating set if there exists a subset U of G satisfying the following conditions:

(a) there is $u_0 \in U$ with $u \leqslant u_0$ for all $u \in U$;

(b) for each compact subset K of G, there is a positive integer n such that

$$K \subset \overbrace{U + \cdots + U}^{n} = \{u_1 + \cdots + u_n : u_i \in U, i = 1, 2, \ldots, n\}.$$

Lemma 1. *Let G have a generating set U and let $\mathscr{G} = \{T_s : s \in G\}$ be a continuous representation of G as nonexpansive mappings from C into itself. Then for each weakly open neighborhood W of $F(\mathscr{G})$, there exists $\varepsilon > 0$ such that $\{x \in C : \sup_{u \in U} \|x - T_u x\| \leqslant \varepsilon\} \subset W$.*

Proof. Suppose that there exist a weakly open neighborhood W of $F(\mathscr{G})$ and a sequence $\{x_n\}$ in $C \backslash W$ such that

$$\sup_{u \in U} \|x_n - T_u x_n\| \leqslant 1/n.$$

Since $C \backslash W$ is weakly sequentially compact, we may assume that x_n converges weakly as $n \to \infty$ to some element x_0 of $C \backslash W$. Then $x_0 \in F(T_u)$ for every $u \in U$ by the demiclosedness of $I - T_u$, where I is the identity mapping. On the other hand, since U is a generating set of G, for each $s \in G$, there exist $u_1, \ldots, u_n \in U$ such that $s = u_1 + \cdots + u_n$. Therefore $T_s x_0 = T_{u_1 + \cdots + u_n} x_0 = T_{u_1} \cdots T_{u_n} x_0 = x_0$. This implies $x_0 \in F(\mathscr{G})$, which contradicts to $x_0 \in C \setminus W$.

LEMMA 2 [9]. *Let μ be a compact mean on G and let A be an equicontinuous subset of $cm(G)$. Then, for every $\varepsilon > 0$, there exists a finite mean λ on G such that*

$$|\mu(f) - \lambda(f)| < \varepsilon$$

for all $f \in A$.

Throughout the rest of this paper, we assume that the topology of G is induced by a metric ρ defined on G such that $\rho(u + v, u + w) \leqslant \rho(v, w)$ for every $u, v, w \in G$.

DEFINITION 2. Let G have a generating set U and let $\mathscr{G} = \{T_s : s \in G\}$ be a continuous representation of G as nonexpansive mappings from C into itself. Then, we call a family Φ of functions φ of G into C a U-family if Φ satisfies the following conditions:

(a) Φ is equi-uniformly continuous, that is, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\rho(s, t) < \delta$ implies $\|\varphi_s - \varphi_t\| < \varepsilon$ for every $\varphi \in \Phi$;

(b) for each $\varepsilon > 0$, there exists $s_0 \in G$ such that

$$\sup_{s_0 \leqslant s} \sup_{u \in U} \|T_u \varphi_s - \varphi_{s+u}\| < \varepsilon$$

for every $\varphi \in \Phi$.

In [5], Bruck proved the following Lemma.

LEMMA 3 [5]. *Let C be a bounded closed convex subset of a uniformly convex Banach space E. Then, there exists a strictly increasing and convex function γ: $[0, \infty) \to [0, \infty)$ with $\gamma(0) = 0$ such that, for every nonexpansive mapping T from C into E, $x, y \in C$ and $0 \leqslant c \leqslant 1$,*

$$\gamma(\|cTx + (1-c)\,Ty - T(cx + (1-c)\,y)\|) \leqslant \|x - y\| - \|Tx - Ty\|.$$

The function γ in Lemma 3 is continuous on $[0, \infty)$. In fact, γ is continuous on $(0, \infty)$, since γ is a convex function. Also, γ is continuous at 0, since

$$0 < \gamma(t) = \gamma((1-t) \cdot 0 + t \cdot 1) \leqslant (1-t)\,\gamma(0) + t\gamma(1) = t\gamma(1) \to 0$$

as $t \to 0$. Furthermore, from continuity and monotonicity of γ, we have

$$\gamma(\sup a_i) = \sup \gamma(a_i), \tag{3.1}$$

where a_i are nonnegative numbers.

Using Lemmas 2 and 3, we obtain the following lemma.

LEMMA 4. *Let G have a generating set U, let $\mathscr{G} = \{T_s : s \in G\}$ be a continuous representation of G as nonexpansive mappings from C into itself, and let Φ be a U-family of functions φ of G into C. Let μ be a compact mean on G and $\varepsilon > 0$. Then, there exists a finite mean λ on G such that*

(a) $\|\mu_s\langle\varphi_{w+s}\rangle - \lambda_s\langle\varphi_{w+s}\rangle\| < \varepsilon$,

(b) $$\left|\int \|\eta_u\langle\varphi_{w+s+u}\rangle - \eta'_u\langle\varphi_{w+s+u}\rangle\|\, d\mu(s) - \int \|\eta_u\langle\varphi_{w+s+u}\rangle - \eta'_u\langle\varphi_{w+s+u}\rangle\|\, d\lambda(s)\right| < \varepsilon,$$

and

(c) $$\left|\int \sup_{u \in U} \|T_u\varphi_{w+s} - \varphi_{w+s+u}\|\, d\mu(s) - \int \sup_{u \in U} \|T_u\varphi_{w+s} - \varphi_{w+s+u}\|\, d\lambda(s)\right| < \varepsilon$$

for every $\varphi \in \Phi$, $w \in G$, and every mean η, η' on G.

Proof. Let $y^* \in E^*$, $\varphi \in \Phi$, $w \in G$ and let η, η' be means on G. Then we define three real valued functions on G by

$$f_{\varphi,w,y^*}(s) = \langle\varphi_{w+s}, y^*\rangle,$$
$$g_{\varphi,w,\eta,\eta'}(s) = \|\eta_u\langle\varphi_{w+s+u}\rangle - \eta'_u\langle\varphi_{w+s+u}\rangle\|,$$

and

$$h_{\varphi,w}(s) = \sup_{u \in U} \|T_u\varphi_{w+s} - \varphi_{w+s+u}\|$$

for each $s \in G$. Put $A = \{f_{\varphi,w,y^*}, g_{\varphi,w,\eta,\eta'}, h_{\varphi,w} : \varphi \in \Phi,\ w \in G,\ y^* \in E^*$ with $\|y^*\| \leqslant 1$ and η, η' are means on $G\}$. Since Φ is a U-family, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\rho(s, t) < \delta$ implies $\|\varphi_{w+s} - \varphi_{w+t}\| < \varepsilon$ for every

$\varphi \in \Phi$ and $w \in G$. Then, for every $s, t \in G$ with $\rho(s, t) < \delta$, we have

$$\begin{aligned} |f_{\varphi,w,y^*}(s) - f_{\varphi,w,y^*}(t)| &= |\langle \varphi_{w+s} - \varphi_{w+t}, y^* \rangle| \\ &\leqslant \|\varphi_{w+s} - \varphi_{w+t}\| \\ &< \varepsilon \end{aligned}$$

and

$$\begin{aligned} &|g_{\varphi,w,\eta,\eta'}(s) - g_{\varphi,w,\eta,\eta'}(t)| \\ &\quad = |\, \|\eta_u\langle \varphi_{w+s+u}\rangle - \eta'_u\langle \varphi_{w+s+u}\rangle\| - \|\eta_u\langle \varphi_{w+t+u}\rangle - \eta'\langle \varphi_{w+t+u}\rangle\| \,| \\ &\quad \leqslant \|\eta_u\langle \varphi_{w+s+u}\rangle - \eta_u\langle \varphi_{w+t+u}\rangle\| + \|\eta'_u\langle \varphi_{w+s+u}\rangle - \eta'_u\langle \varphi_{w+t+u}\rangle\| \\ &\quad \leqslant 2 \cdot \sup_{u \in U} \|\varphi_{w+s+u} - \varphi_{w+t+u}\| \\ &\quad < 2\varepsilon. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} h_{\varphi,w}(t) = \sup_{u \in U} \|T_u \varphi_{w+t} - \varphi_{w+t+u}\| &\geqslant \|T_v \varphi_{w+t} - \varphi_{w+t+v}\| \\ &\geqslant -\|T_v \varphi_{w+t} - T_v \varphi_{w+s}\| - \|\varphi_{w+t+v} - \varphi_{w+s+v}\| \\ &\quad + \|T_v \varphi_{w+s} - \varphi_{w+s+v}\| \\ &> -2\varepsilon + \|T_v \varphi_{w+s} - \varphi_{w+s+v}\| \end{aligned}$$

for every $v \in U$. Therefore, we obtain

$$h_{\varphi,w}(t) \geqslant -2\varepsilon + \sup_{v \in U} \|T_v \varphi_{w+s} - \varphi_{w+s+v}\| = h_{\varphi,w}(s) - 2\varepsilon$$

and hence

$$|h_{\varphi,w}(s) - h_{\varphi,w}(t)| \leqslant 2\varepsilon. \tag{3.2}$$

Thus, A is equicontinuous and hence by Lemma 2, there exists a finite mean λ on G such that

$$\left| \int f_{\varphi,w,y^*}(s)\, d\mu(s) - \int f_{\varphi,w,y^*}(s)\, d\lambda(s) \right| < \varepsilon,$$

$$\left| \int g_{\varphi,w,\eta,\eta'}(s)\, d\mu(s) - \int g_{\varphi,w,\eta,\eta'}(s)\, d\lambda(s) \right| < \varepsilon,$$

and

$$\left| \int h_{\varphi,w}(s)\, d\mu(s) - \int h_{\varphi,w}(s)\, d\lambda(s) \right| < \varepsilon$$

for every f_{φ,w,y^*}, $g_{\varphi,w,\eta,\eta'}$, $h_{\varphi,w} \in A$. Thus, we have

$$\begin{aligned}
&|\langle \mu_s\langle \varphi_{w+s}\rangle - \lambda_s\langle \varphi_{w+s}\rangle, y^*\rangle| \\
&\quad = \left| \int \langle \varphi_{w+s}, y^*\rangle \, d\mu(s) - \int \langle \varphi_{w+s}, y^*\rangle \, d\lambda(s) \right| \\
&\quad = \left| \int f_{\varphi,w,y^*}(s) \, d\mu(s) - \int f_{\varphi,w,y^*}(s) \, d\lambda(s) \right| \\
&\quad < \varepsilon,
\end{aligned}$$

$$\begin{aligned}
&\left| \int \|\eta_u\langle \varphi_{w+s+u}\rangle - \eta'_u\langle \varphi_{w+s+u}\rangle\| \, d\mu(s) \right. \\
&\qquad \left. - \int \|\eta_u\langle \varphi_{w+s+u}\rangle - \eta'_u\langle \varphi_{w+s+u}\rangle\| \, d\lambda(s) \right| \\
&\quad = \left| \int g_{\varphi,w,\eta,\eta'}(s) \, d\mu(s) - \int g_{\varphi,w,\eta,\eta'}(s) \, d\lambda(s) \right| \\
&\quad < \varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
&\left| \int \sup_{u \in U} \|T_u \varphi_{w+s} - \varphi_{w+s+u}\| \, d\mu(s) - \int \sup_{u \in U} \|T_u \varphi_{w+s} - \varphi_{w+s+u}\| \, d\lambda(s) \right| \\
&\quad = \left| \int h_{\varphi,w}(s) \, d\mu(s) - \int h_{\varphi,w}(s) \, d\lambda(s) \right| \\
&\quad < \varepsilon
\end{aligned}$$

for every $\varphi \in \Phi$, $w \in G$, $y^* \in E^*$ with $\|y^*\| \leqslant 1$ and means η, η' on G. Therefore, the conclusion follows.

By using Lemma 4, for a net $\{\mu_\alpha\}$ of compact means on G such that

$$\lim_\alpha \sup_{u \in U} \|\mu_\alpha - r_u^* \mu_\alpha\|_c = 0,$$

and a corresponding net $\{m_\alpha\}$ of positive numbers such that $\lim_\alpha m_\alpha = 0$, we can define a net $\{\lambda_\alpha\}$ of finite means on G such that

$$\text{(a)} \quad \left\| \int \varphi_{w+s} \, d\mu_\alpha(s) - \int \varphi_{w+s} \, d\lambda_\alpha(s) \right\| < m_\alpha, \tag{3.3}$$

$$\begin{aligned}
\text{(b)} \quad &\left| \int \|\eta_t\langle \varphi_{w+s+t}\rangle - \eta'_t\langle \varphi_{w+s+t}\rangle\| \, d\mu_\alpha(s) \right. \\
&\qquad \left. - \int \|\eta_t\langle \varphi_{w+s+t}\rangle - \eta'_t\langle \varphi_{w+s+t}\rangle\| \, d\lambda_\alpha(s) \right| < m_\alpha,
\end{aligned} \tag{3.4}$$

and

$$\text{(c)} \quad \left| \int \sup_{u \in U} \|T_u \varphi_{w+s} - \varphi_{w+s+u}\| \, d\mu_\alpha(s) - \int \sup_{u \in U} \|T_u \varphi_{w+s} - \varphi_{w+s+u}\| \, d\lambda_\alpha(s) \right| < m_\alpha \tag{3.5}$$

for every $\varphi \in \Phi$, $w \in G$, and means η, η' on G. We can also prove the following Lemma for such a net $\{\lambda_\alpha\}$ of finite means on G.

LEMMA 5. *Let λ be a finite mean on G. Then,*

$$\lim_\alpha \sup_{\varphi \in \Phi} \int \sup_{u \in U} \|T_u(\lambda_t \langle \varphi_{s+t} \rangle) - \lambda_t \langle \varphi_{s+t+u} \rangle\| \, d\lambda_\alpha(s) = 0.$$

Proof. We shall prove by mathematical induction. Let $\lambda = \delta_{t_1}$, $t_1 \in G$. For every $\varepsilon > 0$, there exists $s_0 \in G$ from Definition 2 such that $\sup_{u \in U} \|T_u \varphi_s - \varphi_{s+u}\| < \varepsilon$ for every $\varphi \in \Phi$ and $s \geqslant s_0$. On the other hand, since $s_0 = u_1 + \cdots + u_n$ for some $u_i \in U$, $i = 1, 2, \ldots, n$, we obtain

$$\begin{aligned} \|\mu_\alpha - r_{s_0}^* \mu_\alpha\|_c &\leqslant \|\mu_\alpha - r_{u_1}^* \mu_\alpha\|_c + \|r_{u_1}^* \mu_\alpha - r_{u_1+u_2}^* \mu_\alpha\|_c \\ &\quad + \cdots + \|r_{u_1+\cdots+u_{n-1}}^* \mu_\alpha - r_{s_0}^* \mu_\alpha\|_c \\ &\leqslant n \cdot \sup_{u \in U} \|\mu_\alpha - r_u^* \mu_\alpha\|_c. \end{aligned}$$

Therefore, there exists α_0 such that $m_\alpha < \varepsilon$ and $\|\mu_\alpha - r_{s_0}^* \mu_\alpha\|_c < \varepsilon/D$ for every $\alpha \geqslant \alpha_0$, where D denotes the diameter of C. Then, from (3.2) and (3.5),

$$\begin{aligned} &\sup_{\varphi \in \Phi} \int \sup_{u \in U} \|T_u \varphi_{s+t_1} - \varphi_{s+t_1+u}\| \, d\lambda_\alpha(s) \\ &\quad \leqslant \sup_{\varphi \in \Phi} \int \sup_{u \in U} \|T_u \varphi_{s+t_1} - \varphi_{s+t_1+u}\| \, d\mu_\alpha(s) + m_\alpha \\ &\quad < \sup_{\varphi \in \Phi} \left\{ \int \left(\sup_{u \in U} \|T_u \varphi_{s+t_1} - \varphi_{s+t_1+u}\| \right. \right. \\ &\qquad \left. - \sup_{u \in U} \|T_u \varphi_{s_0+s+t_1} - \varphi_{s_0+s+t_1+u}\| \right) d\mu_\alpha(s) \\ &\qquad \left. + \int \sup_{u \in U} \|T_u \varphi_{s_0+s+t_1} - \varphi_{s_0+s+t_1+u}\| \, d\mu_\alpha(s) \right\} + \varepsilon \\ &\quad \leqslant \sup_{\varphi \in \Phi} \int \sup_{u \in U} \|T_u \varphi_{s+t_1} - \varphi_{s+t_1+u}\| \, d(\mu_\alpha - r_{s_0}^* \mu_\alpha)(s) + 2\varepsilon \\ &\quad \leqslant D \cdot \|\mu_\alpha - r_{s_0}^* \mu_\alpha\|_c + 2\varepsilon < 3\varepsilon \end{aligned}$$

for every $\alpha \geqslant \alpha_0$. Therefore, we obtain

$$\lim_{\alpha} \sup_{\varphi \in \Phi} \int \sup_{u \in U} \| T_u \lambda_t \langle \varphi_{s+t} \rangle - \lambda_t \langle \varphi_{s+t+u} \rangle \| \, d\lambda_\alpha(s) = 0$$

for $\lambda = \delta_{t_1}$.

Suppose that

$$\lim_{\alpha} \sup_{\varphi \in \Phi} \int \sup_{u \in U} \| T_u \lambda_t \langle \varphi_{s+t} \rangle - \lambda_t \langle \varphi_{s+t+u} \rangle \| \, d\lambda_\alpha(s) = 0$$

for each $\lambda = \sum_{i=1}^{n-1} a_i \delta_{t_i}$ $(a_i \geqslant 0,\ t_i \in G,\ i = 1, 2, ..., n-1,\ \sum_{i=1}^{n-1} a_i = 1)$. Let $\lambda = \sum_{i=1}^{n} a_i \delta_{t_i}$ $(a_i > 0,\ t_i \in G,\ i = 1, 2, ..., n,\ \sum_{i=1}^{n} a_i = 1)$ and $\varepsilon > 0$. Define $\eta = (\sum_{i=1}^{n-1} a_i \delta_{t_i})/(1 - a_n)$. Then,

$$\begin{aligned}
&\| T_u \lambda_t \langle \varphi_{s+t} \rangle - \lambda_t \langle \varphi_{s+t+u} \rangle \| \\
&\quad = \| T_u \lambda_t \langle \varphi_{s+t} \rangle - (1 - a_n)\, \eta_t \langle \varphi_{s+t+u} \rangle - a_n \varphi_{s+t_n+u} \| \\
&\quad \leqslant \| T_u \lambda_t \langle \varphi_{s+t} \rangle - (1 - a_n)\, T_u \eta_t \langle \varphi_{s+t} \rangle - a_n T_u \varphi_{s+t_n} \| \\
&\qquad + (1 - a_n)\, \| T_u \eta_t \langle \varphi_{s+t} \rangle - \eta_t \langle \varphi_{s+t+u} \rangle \| \\
&\qquad + a_n \cdot \| T_u \varphi_{s+t_n} - \varphi_{s+t_n+u} \|
\end{aligned} \tag{3.6}$$

for every $u \in U, s \in G$, and $\varphi \in \Phi$. On the other hand, by (3.1) and Lemma 3, we have for every $s \in G$ and $\varphi \in \Phi$,

$$\begin{aligned}
&\gamma \big(\sup_{u \in U} \| T_u \lambda_t \langle \varphi_{s+t} \rangle - (1 - a_n)\, T_u \eta_t \langle \varphi_{s+t} \rangle - a_n T_u \varphi_{s+t_n} \| \big) \\
&\quad = \sup_{u \in U} \gamma (\| T_u ((1 - a_n)\, \eta_t \langle \varphi_{s+t} \rangle + a_n \varphi_{s+t_n}) \\
&\qquad - (1 - a_n)\, T_u \eta_t \langle \varphi_{s+t} \rangle - a_n T_u \varphi_{s+t_n} \|) \\
&\quad \leqslant \sup_{u \in U} (\| \eta_t \langle \varphi_{s+t} \rangle - \varphi_{s+t_n} \| - \| T_u \eta_t \langle \varphi_{s+t} \rangle - T_u \varphi_{s+t_n} \|) \\
&\quad = \| \eta_t \langle \varphi_{s+t} \rangle - \varphi_{s+t_n} \| - \| T_{u_0} \eta_t \langle \varphi_{s+t} \rangle - T_{u_0} \varphi_{s+t_n} \| \\
&\quad \leqslant \| \eta_t \langle \varphi_{s+t} \rangle - \varphi_{s+t_n} \| - \| \eta_t \langle \varphi_{s+t+u_0} \rangle - \varphi_{s+t_n+u_0} \| \\
&\qquad + \| \eta_t \langle \varphi_{s+t+u_0} \rangle - T_{u_0} \eta_t \langle \varphi_{s+t} \rangle \| + \| \varphi_{s+t_n+u_0} - T_{u_0} \varphi_{s+t_n} \|,
\end{aligned} \tag{3.7}$$

where u_0 is the fixed element of U assumed in Definition 1. Since each λ_α is a finite mean on G and γ is convex, from (3.4) and (3.7), we obtain

$$\gamma\left(\int \sup_{u\in U} \|T_u\lambda_t\langle\varphi_{s+t}\rangle-(1-a_n)\,T_u\eta_t\langle\varphi_{s+t}\rangle-a_nT_u\varphi_{s+t_n}\|\,d\lambda_\alpha(s)\right)$$

$$\leqslant\int \gamma\,(\sup_{u\in U}\|T_u\lambda_t\langle\varphi_{s+t}\rangle-(1-a_n)\,T_u\eta_t\langle\varphi_{s+t}\rangle-a_nT_u\varphi_{s+t_n}\|)\,d\lambda_\alpha(s)$$

$$\leqslant\int (\|\eta_t\langle\varphi_{s+t}\rangle-\varphi_{s+t_n}\|-\|\eta_t\langle\varphi_{s+t+u_0}\rangle-\varphi_{s+t_n+u_0}\|$$

$$+\|\eta_t\langle\varphi_{s+t+u_0}\rangle-T_{u_0}\eta_t\langle\varphi_{s+t}\rangle\|+\|\varphi_{s+t_n+u_0}-T_{u_0}\varphi_{s+t_n}\|)\,d\lambda_\alpha(s)$$

$$\leqslant\int \|\eta_t\langle\varphi_{s+t}\rangle-\varphi_{s+t_n}\|\,d\mu_\alpha(s)$$

$$-\int \|\eta_t\langle\varphi_{s+t+u_0}\rangle-\varphi_{s+t_n+u_0}\|\,d\mu_\alpha(s)$$

$$+\int (\|\eta_t\langle\varphi_{s+t+u_0}\rangle-T_{u_0}\eta_t\langle\varphi_{s+t}\rangle\|$$

$$+\|\varphi_{s+t_n+u_0}-T_{u_0}\varphi_{s+t_n}\|)\,d\lambda_\alpha(s)+2m_\alpha$$

$$=\int \|\eta_t\langle\varphi_{s+t}\rangle-\varphi_{s+t_n}\|\,d(\mu_\alpha-r^*_{u_0}\mu_\alpha)(s)$$

$$+\int (\|\eta_t\langle\varphi_{s+t+u_0}\rangle-T_{u_0}\eta_t\langle\varphi_{s+t}\rangle\|$$

$$+\|\varphi_{s+t_n+u_0}-T_{u_0}\varphi_{s+t_n}\|)\,d\lambda_\alpha(s)+2m_\alpha$$

$$\leqslant\sup_{s\in G}\|\eta_t\langle\varphi_{s+t}\rangle-\varphi_{s+t_n}\|\cdot\|\mu_\alpha-r^*_{u_0}\mu_\alpha\|_c$$

$$+\int \sup_{u\in U}(\|\eta_t\langle\varphi_{s+t+u}\rangle-T_u\eta_t\langle\varphi_{s+t}\rangle\|$$

$$+\|\varphi_{s+t_n+u}-T_u\varphi_{s+t_n}\|)\,d\lambda_\alpha(s)+2m_\alpha \tag{3.8}$$

for every $\varphi\in\Phi$. Now, from the assumptions, for each $\varepsilon>0$, we can choose α_0 such that

$$m_\alpha<\varepsilon,$$

$$\|\mu_\alpha-r^*_{u_0}\mu_\alpha\|_c<\varepsilon/D,$$

$$\sup_{\varphi\in\Phi}\int \sup_{u\in U}\|T_u\varphi_{s+t_n}-\varphi_{s+t_n+u}\|\,d\lambda_\alpha(s)<\varepsilon,$$

and

$$\sup_{\varphi \in \Phi} \int \sup_{u \in U} \| T_u \eta_t \langle \varphi_{s+t} \rangle - \eta_t \langle \varphi_{s+t+u} \rangle \| \, d\lambda_\alpha(s) < \varepsilon$$

for every $\alpha \geqslant \alpha_0$. Then, with (3.6) and (3.8), we obtain

$$\begin{aligned}
&\int \sup_{u \in U} \| T_u \lambda_t \langle \varphi_{s+t} \rangle - \lambda_t \langle \varphi_{s+t+u} \rangle \| \, d\lambda_\alpha(s) \\
&\quad \leqslant \int \sup_{u \in U} \| T_u \lambda_t \langle \varphi_{s+t} \rangle - (1-a_n) \, T_u \eta_t \langle \varphi_{s+t} \rangle - a_n T_u \varphi_{s+t_n} \| \, d\lambda_\alpha(s) \\
&\qquad + (1-a_n) \int \sup_{u \in U} \| T_u \eta_t \langle \varphi_{s+t} \rangle - \eta_t \langle \varphi_{s+t+u} \rangle \| \, d\lambda_\alpha(s) \\
&\qquad + a_n \int \sup_{u \in U} \| T_u \varphi_{s+t_n} - \varphi_{s+t_n+u} \| \, d\lambda_\alpha(s) \\
&\quad \leqslant \gamma^{-1} \Big\{ \sup_{s \in G} \| \eta_t \langle \varphi_{s+t} \rangle - \varphi_{s+t_n} \| \cdot \| \mu_\alpha - r_{u_0}^* \mu_\alpha \|_c \\
&\qquad + \int \sup_{u \in U} (\| \eta_t \langle \varphi_{s+t+u} \rangle - T_u \eta_t \langle \varphi_{s+t} \rangle \| \\
&\qquad + \| \varphi_{s+t_n+u} - T_u \varphi_{s+t_n} \|) \, d\lambda_\alpha(s) + 2m_\alpha \Big\} \\
&\qquad + (1-a_n) \int \sup_{u \in U} \| T_u \eta_t \langle \varphi_{s+t} \rangle - \eta_t \langle \varphi_{s+t+u} \rangle \| \, d\lambda_\alpha(s) \\
&\qquad + a_n \int \sup_{u \in U} \| T_u \varphi_{s+t_n} - \varphi_{s+t_n+u} \| \, d\lambda_\alpha(s) \\
&\quad < \gamma^{-1}(5\varepsilon) + \varepsilon
\end{aligned}$$

for all $\alpha \geqslant \alpha_0$ and $\varphi \in \Phi$. Since γ^{-1} is continuous, the proof is completed.

4. Main Results

In this section, we study the asymptotic behavior of orbits of semigroups of nonexpansive mappings in Banach spaces.

Theorem 1. *Let G have a generating set U, let $\mathscr{G} = \{T_t : t \in G\}$ be a continuous representation of G as nonexpansive mappings from C into itself, and let Φ be a U-family of functions of G into C. Let $\{\mu_\alpha\}$ be a net of compact means on G such that* $\lim_\alpha \sup_{u \in U} \| \mu_\alpha - r_u^* \mu_\alpha \|_c = 0$ *and* $\{m_\alpha\}$ *be a* *corresponding net of positive numbers with* $\lim_\alpha m_\alpha = 0$. *Then for a weak neighborhood* W *of* $F(\mathscr{G})$, *there exists* α_0 *such that* $\int \varphi_{w+s}\, d\mu_\alpha(s) \in W$ *for every* $\alpha \geqslant \alpha_0$, $w \in G$, *and* $\varphi \in \Phi$.

Proof. We have only to show $\int \varphi_s\, d\mu_\alpha(s) \in W$ for every $\varphi \in \Phi$ and sufficiently large α. In fact, let $\Phi' = \{r_\omega \varphi : \varphi \in \Phi, w \in G\}$. It is easy to see that Φ' is a U-family. Then, the conclusion follows from $\int \varphi_s\, d\mu_\alpha(s) \in W$ for every $\varphi \in \Phi'$ and sufficiently large α. Since E is uniformly convex, $F(\mathscr{G})$ is a closed convex subset of C and therefore weakly compact. Then, easy topological argument shows the existence of a convex weakly open neighborhood W' of $F(G)$ and $\delta > 0$ such that $W' + B_\delta \subset W$, where $B_\delta = \{x \in E : \|x\| \leqslant \delta\}$. Let $D = \sup_{x \in C} \|x\|$. Then $\operatorname{diam}(C) \leqslant 2D$. Using Lemma 1, choose $\varepsilon > 0$ so small that

$$\varepsilon + \varepsilon^2/4 < \delta/2,$$

$$2\varepsilon D < \delta/2,$$

and

$$\{x \in C : \sup_{u \in U} \|x - T_u x\| \leqslant \varepsilon\} \subset W'.$$

By the assumption of Theorem, there exists α_0 such that

$$\sup_{u \in U} \|\mu_{\alpha_0} - r_u^* \mu_{\alpha_0}\|_c < \varepsilon^2/(4D) \tag{4.1}$$

and

$$m_\alpha < \varepsilon^2/4$$

for every $\alpha \geqslant \alpha_0$. Hence, if we choose $\{\lambda_\alpha\}$ in Lemma 5, we have

$$\left\| \int \varphi_{s+t}\, d\mu_\alpha(t) - \int \varphi_{s+t}\, d\lambda_\alpha(t) \right\| < \varepsilon^2/4 \tag{4.2}$$

for every $\alpha \geqslant \alpha_0$, $s \in G$, and $\varphi \in \Phi$. Here we put $\mu' = \mu_{\alpha_0}$ and $\lambda' = \lambda_{\alpha_0}$. Then, for every $u \in U$, $s \in G$, and $\varphi \in \Phi$,

$$\begin{aligned}
&\|\lambda'_t\langle \varphi_{s+t}\rangle - \lambda'_t\langle \varphi_{s+t+u}\rangle\| \\
&\quad \leqslant \|\lambda'_t\langle \varphi_{s+t}\rangle - \mu'_t\langle \varphi_{s+t}\rangle\| + \|\mu'_t\langle \varphi_{s+t}\rangle - \mu'_t\langle \varphi_{s+t+u}\rangle\| \\
&\qquad + \|\mu'_t\langle \varphi_{s+t+u}\rangle - \lambda'_t\langle \varphi_{s+t+u}\rangle\| \\
&\quad \leqslant \varepsilon^2/2 + \left\| \int \varphi_{s+t}\, d\mu'(t) - \int \varphi_{s+t}\, d(r_u^*\mu')(t) \right\| \\
&\quad \leqslant \varepsilon^2/2 + \sup_{t \in G} \|\varphi_t\| \cdot \|\mu' - r_u^*\mu'\|_c \\
&\quad < 3\varepsilon^2/4.
\end{aligned} \tag{4.3}$$

Using Lemma 5 for λ', we have $\alpha_1 \geqslant \alpha_0$ such that

$$\int \sup_{u \in U} \|T_u \lambda'_t \langle \varphi_{s+t} \rangle - \lambda'_t \langle \varphi_{s+t+u} \rangle\| \, d\lambda_\alpha(s) < \varepsilon^2/4 \tag{4.4}$$

for every $\alpha \geqslant \alpha_1$ and $\varphi \in \Phi$. Thus, from (4.3) and (4.4), we obtain

$$\int \sup_{u \in U} \|\lambda'_t \langle \varphi_{s+t} \rangle - T_u \lambda'_t \langle \varphi_{s+t} \rangle\| \, d\lambda_\alpha(s) < \varepsilon^2 \tag{4.5}$$

for every $\alpha \geqslant \alpha_1$ and $\varphi \in \Phi$. Suppose $\lambda' = \sum_{i=1}^n a_i \delta_{t_i}$ $(t_i \in G, a_i \geqslant 0, i = 1, 2,..., n, \sum_{i=1}^n a_i = 1)$. Let k be a positive integer such that

$$\{t_i : i = 1, 2,..., n\} \subset \overbrace{U + U + \cdots + U}^{k}.$$

Then there exists $\alpha_2 \geqslant \alpha_1$ such that

$$\sup_{u \in U} \|\mu_\alpha - r_u^* \mu_\alpha\|_c < \varepsilon/(kD) \qquad \text{for every } \alpha \geqslant \alpha_2. \tag{4.6}$$

Fix $\alpha \geqslant \alpha_2$ and $\varphi \in \Phi$ arbitrarily. Assume $\lambda_\alpha = \sum_{j=1}^m b_j \delta_{s_j}$ $(s_j \in G, b_j > 0, j = 1, 2,..., m, \sum_{j=1}^m b_j = 1)$ and let $S = \{s_j : j = 1, 2,..., m\}$. Put

$$A = \{s \in S : \sup_{u \in U} \|\lambda'_t \langle \varphi_{s+t} \rangle - T_u \lambda'_t \langle \varphi_{s+t} \rangle\| > \varepsilon\}$$

and

$$B = S - A.$$

If $\lambda_\alpha(1_A) > \varepsilon$, then, using (4.5),

$$\begin{aligned} \varepsilon^2 &> \int \sup_{u \in U} \|\lambda'_t \langle \varphi_{s+t} \rangle - T_u \lambda'_t \langle \varphi_{s+t} \rangle\| \, d\lambda_\alpha(s) \\ &\geqslant \int \varepsilon \cdot 1_A(s) \, d\lambda_\alpha(s) \\ &> \varepsilon^2, \end{aligned}$$

which is a contradiction. Thus, $\lambda_\alpha(1_A) \leqslant \varepsilon$. Now, fix $f \in F(G)$ and define

$$x_1 = \sum \{b_j \cdot f : 1 \leqslant j \leqslant m, s_j \in A\} + \sum \{b_j \cdot \lambda'_t \langle \varphi_{s_j+t} \rangle : 1 \leqslant j \leqslant m, s_j \in B\}$$

and

$$x_2 = \sum \{b_j \cdot (\lambda'_t \langle \varphi_{s_j+t} \rangle - f) : 1 \leqslant j \leqslant m, s_j \in A\}.$$

Then, since $\int \varphi_{s_j+t}\, d\lambda'(t) \in W'$ for every $s_j \in B$, $f \in W'$, and W' is convex, we obtain $x_1 \in W'$. We also obtain $\|x_2\| \leqslant 2\varepsilon D$ by $\sum \{b_j : 1 \leqslant j \leqslant m, s_j \in A\} = \lambda_\alpha(1_A) \leqslant \varepsilon$. Thus

$$\begin{aligned}\int \lambda'_t\langle\varphi_{s+t}\rangle\, d\lambda_\alpha(s) &= x_1 + x_2 \\ &\in W' + B_{2\varepsilon D} \\ &\subset W' + B_{\delta/2}.\end{aligned}$$

While, using (4.2) and (4.6),

$$\begin{aligned}&\left|\int \varphi_s\, d\mu_\alpha(s) - \int \lambda'_t\langle\varphi_{s+t}\rangle\, d\lambda_\alpha(s)\right| \\ &\leqslant \left|\int \varphi_s\, d\mu_\alpha(s) - \int \sum_{i=1}^{n} a_i \varphi_{s+t_i}\, d\mu_\alpha(s)\right| \\ &\quad + \left|\int \sum_{i=1}^{n} a_i \varphi_{s+t_i}\, d\mu_\alpha(s) - \int \sum_{i=1}^{n} a_i \varphi_{s+t_i}\, d\lambda_\alpha(s)\right| \\ &\leqslant \sum_{i=1}^{n} a_i \left|\int \varphi_s\, d\mu_\alpha(s) - \int \varphi_s\, d(r^*_{t_i}\mu_\alpha)(s)\right| \\ &\quad + \sum_{i=1}^{n} a_i \cdot \left|\int \varphi_{s+t_i}\, d\mu_\alpha(s) - \int \varphi_{s+t_i}\, d\lambda_\alpha(s)\right| \\ &< D \cdot \sup_{1 \leqslant i \leqslant n} \|\mu_\alpha - r^*_{t_i}\mu_\alpha\|_c + \sum_{i=1}^{n} a_i \varepsilon^2/4 \\ &< D \cdot k \cdot \sup_{u \in U} \|\mu_\alpha - r^*_u \mu_\alpha\|_c + \varepsilon^2/4 \\ &< \varepsilon + \varepsilon^2/4 \\ &< \delta/2.\end{aligned}$$

Therefore, since $\alpha \geqslant \alpha_2$ and $\varphi \in \Phi$ are arbitrary, $\int \varphi_s\, d\mu_\alpha(s) \in W' + B_{\delta/2} + B_{\delta/2} \subset W$ for every $\alpha \geqslant \alpha_2$ and $\varphi \in \Phi$.

By using Theorem 1, we have the following theorem.

THEOREM 2. *Under the same assumption in Theorem* 1 *for* G, $\mathscr{G} = \{T_t : t \in G\}$, $\{\mu_\alpha\}$, *and* $\{m_\alpha\}$, *let* $\{x_t : t \in G\}$ *be a net in* C *such that* $t \to x_t$ *is uniformly continuous and* $\lim_t \sup_{u \in U} \|x_{t+u} - T_u x_t\| = 0$. *Then, for every weak neighborhood* W *of* $F(G)$, *there exists* α_0 *such that* $\int x_{k+s}\, d\mu_\alpha(s) \in W$ *for every* $\alpha \geqslant \alpha_0$ *and* $k \in G$.

Proof. Let $\Phi = \{x: t \to x_t, t \in G\}$. Then, Φ is a U-family. Hence the conclusion follows from Theorem 1 directly.

COROLLARY 1 (Bruck [5]). *Assume that S is a nonexpansive mapping on C. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in C such that $\lim_{n\to\infty} \|x_{n+1} - Sx_n\| = 0$. Then, for every weak neighborhood W of $F(S)$, $(1/n)\sum_{i=0}^{n-1} x_{i+k} \in W$ for every $k \geqslant 0$ and sufficiently large n.*

Proof. Put

$$G = \{0, 1, \ldots\} \text{ with metric } \rho \text{ defined by } \rho(m, n) = |m - n| \text{ for all } m, n \in G,$$

$$U = \{0, 1\},$$

$$\{T_t : t \in G\} = \{S^i : i = 0, 1, \ldots\},$$

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta_i, \qquad n = 1, 2, \ldots,$$

and

$$m_n = 1/n, \qquad n = 1, 2, \ldots.$$

Then, Corollary 1 is obvious from Theorem 2.

Here, we shall remember Bruck's remark such that the Cesàro mean $(1/n)\sum_{i=0}^{n-1} x_{i+k}$ need not converge even in one-dimensional cases.

COROLLARY 2. *Assume that $\{S(t): t \geqslant 0\}$ is a one parameter semigroup on C. Let $\{x_s : s \geqslant 0\}$ be a net in C such that $s \to x_s$ is uniformly continuous and $\lim_{s\to\infty} \sup_{0\leqslant t\leqslant 1} \|x_{s+t} - S(t)x_s\| = 0$. Then, for every weak neighborhood W of the set of common fixed points of $\{S(t): t \geqslant 0\}$, $(1/n)\int_0^n x_{k+s}\, ds \in W$ for every $k \geqslant 0$ and sufficiently large n.*

Proof. Put

$$G = [0, \infty) \text{ with usual metric},$$

$$U = [0, 1],$$

$$\{T_t : t \in G\} = \{S(t) : t \geqslant 0\},$$

μ_n be a Hahn–Banach extension of $\psi_n : L^1_{\mathrm{loc}}(G) \cap m(G) \to R$

defined by $\psi_n(f) = \dfrac{1}{n}\displaystyle\int_0^n f(s)\, ds$

for every $f \in L^1_{\mathrm{loc}}(G) \cap m(G)$, $n > 0$,

and

$$m_n = 1/n, \qquad n > 0.$$

Here fix $u \in [0, 1]$ and $f \in cm(G)$ with $\|f\| = 1$ arbitrarily. Then,

$$\begin{aligned}
|\langle \mu_n - r_u^* \mu_n, f \rangle| &= \frac{1}{n} \left| \int_0^n f(s)\, ds - \int_0^n f(s+u)\, ds \right| \\
&= \frac{1}{n} \left| \int_0^n f(s)\, ds - \int_u^{n+u} f(s)\, ds \right| \\
&\leqslant \frac{1}{n} \left(\int_0^u |f(s)|\, ds + \int_n^{n+u} |f(s)|\, ds \right) \\
&\leqslant \frac{1}{n} \cdot 2u \cdot \sup_{0 \leqslant s} |f(s)| \\
&\leqslant \frac{2}{n}.
\end{aligned}$$

This implies that $\lim_n \sup_{0 \leqslant u \leqslant 1} \|\mu_n - r_u^* \mu_n\|_c = 0$. Therefore, Corollary 2 is obvious from Theorem 2.

By using the same method of Theorem 1, we obtain the following ergodic theorem which insures the uniform convergence in initial values.

THEOREM 3. *Under the same assumption in Theorem* 1 *for* $G, \mathscr{G} = \{T_t : t \in G\}$, $\{\mu_\alpha\}$, *and* $\{m_\alpha\}$, *let* C' *be a subset of* C *such that for each* $t \in G$, $s \to t$ *in* G *implies* $T_s x \to T_t x$ *uniformly in* $x \in C'$. *Then, for every weak neighborhood* W *of* $F(G)$, *there exists* α_0 *such that* $\mathscr{T}_{\mu_\alpha} T_t x \in W$ *for every* $\alpha \geqslant \alpha_0$, $t \in G$, *and* $x \in C'$. *Especially, if* $F(G)$ *is singleton, then* $\mathscr{T}_{\mu_\alpha} T_t x$ *converges weakly to the common fixed point uniformly in* $t \in G$ *and* $x \in C'$.

Proof. Define $\varphi^x : s \to T_s x$ for every $s \in G$ and $x \in C'$. Let $\Phi = \{\varphi^x : x \in C'\}$. Then, for each $t \in G$,

$$s \to t \text{ in } G \text{ implies } \varphi^x_{s+w} \to \varphi^x_{t+w} \text{ uniformly in } w \in G \text{ and } \varphi^x \in \Phi \tag{4.7}$$

and

$$\sup_{u \in U} \|T_u \varphi^x_s - \varphi^x_{s+u}\| = 0 \qquad \text{for every } w \in G \text{ and } \varphi^x \in \Phi. \tag{4.8}$$

From conditions (4.7) and (4.8), we easily obtain the same conclusions of Lemmas 4 and 5. Therefore the same proof of Theorem 1 is available to obtain $\mathscr{T}_{\mu_\alpha} T_t x = \int T_s T_t x\, d\mu_\alpha(s) = \int \varphi^x_{s+t}\, d\mu_\alpha(s) \in W$ for every $t \in G$ and $x \in C'$. The last statement of Theorem 3 is obvious.

The following lemma is a slight modification of Proposition proved by Hirano, Kido, and Takahashi [9].

LEMMA 6 [9]. *Let C be a bounded closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm. Let G be a commutative semitopological semigroup, $\mathscr{G} = \{T_t : t \in G\}$ a continuous representation of G as nonexpansive mappings from C into C. Then for $x \in C$, the set $\bigcap_s \overline{\text{co}}\, \{T_t x : t \geqslant s\} \cap F(G)$ consists of at most one point.*

Using Theorem 3 and Lemma 6, we can prove the following general ergodic theorem, which generalizes results of Bruck [5] and Reich [13].

THEOREM 4. *Let C be a bounded closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm, G a commutative semitopological semigroup, and $\mathscr{G} = \{T_t : t \in G\}$ a continuous representation of G as nonexpansive mappings from C into C. Let $\{\mu_\alpha\}$ be a net of compact means on G such that*

$$\lim_\alpha \sup_{u \in U} \|\mu_\alpha - r_u^* \mu_\alpha\|_c = 0$$

and $\{m_\alpha\}$ a corresponding net of positive numbers such that $\lim_\alpha m_\alpha = 0$. Then, for every $x \in C$, $\mathscr{T}_{\mu_\alpha} T_w x$ converges weakly to a common fixed point of $\{T_t : t \in G\}$ uniformly in $w \in G$.

Proof. Fix $x \in C$ and put $x_\alpha = \mathscr{T}_{\mu_\alpha} x$. Let $\{x_{\alpha_\beta}\}_\beta$ be any weakly converging subnet of $\{x_\alpha\}$, say, weak-$\lim_\beta x_{\alpha_\beta} = y$. Then, from Theorem 3, we have $y \in F(G)$. Furthermore, we have $y \in \bigcap_s \overline{\text{co}}\, \{T_t x : t \geqslant s\}$ (see [9]). Thus, we obtain $y \in \bigcap_s \overline{\text{co}}\, \{T_t x : t \geqslant s\} \cap F(G)$. By Lemma 6, y is independent of the choice of a subnet of $\{x_\alpha\}$. Since C is weakly compact, we obtain that $x_\alpha = \mathscr{T}_{\mu_\alpha} x$ itself converges weakly to $y \in F(G) \cap \bigcap_s \overline{\text{co}}\, \{T_t x : t \geqslant s\}$.

Here, we deny the uniform convergence of $\mathscr{T}_{\mu_\alpha} T_w x$, i.e., there exist $y^* \in E^*$, $\varepsilon > 0$ such that for each index α there are $\beta_\alpha \geqslant \alpha$ and $t_\alpha \in G$ with $|\langle \mathscr{T}_{\mu_{\beta_\alpha}} T_{t_\alpha} x - y, y^* \rangle| \geqslant \varepsilon$. Let $\eta_\alpha = r_{t_\alpha}^* \mu_{\beta_\alpha}$ for each α. Then, since G is commutative and $\beta_\alpha \geqslant \alpha$,

$$\begin{aligned} \sup_{u \in U} \|r_u^* \eta_\alpha - \eta_\alpha\|_c &= \sup_{u \in U} \|r_{t_\alpha}^* (r_u^* \mu_{\beta_\alpha} - \mu_{\beta_\alpha})\|_c \\ &\leqslant \sup_{u \in U} \|r_u^* \mu_{\beta_\alpha} - \mu_{\beta_\alpha}\|_c \to 0, \end{aligned}$$

as α tends to infinity. This implies, with the first half of the proof of this theorem, that $\mathscr{T}_{\mu_{\beta_\alpha}} T_{t_\alpha} x = \mathscr{T}_{\eta_\alpha} x$ converges weakly to y. This is a contradiction.

As direct consequences of Theorems 3 and 4, we obtain the following corollaries.

COROLLARY 3 [5, 13]. *Let S be a nonexpansive mapping on a bounded closed convex subset C of a uniformly convex Banach space E. Then, for every weak neighborhood W of* $F(S)$, $(1/n)\sum_{i=1}^{n-1} S^i x \in W$ *for every* $x \in C$ *and sufficiently large n. Furthermore, if E has a Fréchet differentiable norm, then, for each* $x \in C$, $(1/n)\sum_{i=0}^{n-1} S^{i+k}x$ *converges weakly to a fixed point of S uniformly in k.*

COROLLARY 4. *Let* $\{S(t): t \geqslant 0\}$ *be a one parameter semigroup on a bounded closed convex subset C of a uniformly convex Banach space E. Let C′ be a subset of C such that* $S(s)x$ *converges to x uniformly in* $x \in C'$ *as* $s \to 0$. *Then, for every weak niehgborhood W of the set of common fixed points of* $\{S(t), t \geqslant 0\}$, $(1/n)\int_0^n S(s+t)\,x\,ds \in W$ *for every* $x \in C'$, $t \geqslant 0$, *and sufficiently large n. Furthermore if E has a Fréchet differentiable norm, then, for each* $x \in C$, $(1/n)\int_0^n S(s+t)\,x\,ds$ *converges weakly to a common fixed point of* $\{S(s): s \geqslant 0\}$ *uniformly in* $t \geqslant 0$.

Finally, we discuss the convergence of x_t itself.

THEOREM 5. *Under the same assumption in Theorem 1 for* $G, \mathscr{G} = \{T_t: t \in G\}$, $\{\mu_\alpha\}$, *and* $\{m_\alpha\}$, *let* $\{x_t: t \in G\}$ *be a net in C such that* $x_t - T_s x_t$ *converges weakly to 0 for each* $s \in G$. *Furthermore, assume that* (1) $T_s x_t \to T_{s'} x_t$, *as* $s \to s'$ *in G, uniformly in* $t \in G$, *or* (2) $t \to x_t$ *is uniformly continuous and* $\lim_t \sup_{u \in U} \|T_u x_t - x_{t+u}\| = 0$. *Then, for every weak neighborhood W of* $F(G)$, *there exists* t_0 *such that* $x_t \in W$ *for every* $t \geqslant t_0$.

Proof. Deny the assertion. Then there exist a weak neighborhood W of $F(G)$ and $k_t \geqslant t$ for every t, such that $x_{k_t} \notin W$. Select $\delta > 0$, closed convex weak neighborhoods W' of $F(G)$, and V of 0 such that $W' + V + B_\delta \subset W$ as in Theorem 1. Then for each t, by the Hahn–Banach theorem, there exists $y_t^* \in E^*$ with $\|y_t^*\| \leqslant 1$ such that

$$\langle x_{k_t}, y_t^* \rangle > \sup\{\langle w + v, y_t^* \rangle + \delta: w \in W', v \in V\}.$$

On the other hand, let $\varphi^t(s) = T_s x_t$ [resp. $\varphi^t(s) = x_{s+t}$] for every $s, t \in G$, and $\Phi = \{\varphi^t: t \in G\}$. Then, from Theorem 2 or 3, for weak neighborhood W of $F(G)$, there exists α_0 such that

$$\left\langle \int \varphi_s \, d\mu_\alpha(s), y^* \right\rangle \leqslant \sup_{w \in W'} \langle w, y^* \rangle$$

for every $\alpha \geqslant \alpha_0$, $\varphi \in \Phi$, and $y^* \in E^*$ with $\|y^*\| \leqslant 1$. Using Lemma 4 for the compact mean μ_{α_0} on G, select a finite mean $\lambda = \sum_{i=1}^n a_i \delta_{s_i}$ on G such that $\|\int \varphi_s \, d\mu_{\alpha_0}(s) - \int \varphi_s \, d\lambda(s)\| < \delta/2$ for every $\varphi \in \Phi$. Then there exists, from the assumption, $t_0 \in G$ such that $x_t - T_{s_i} x_t \in V$ for every $t \geqslant t_0$ and $i = 1, 2, \ldots, n$.

Hence, there exists $t_1 \in G$ such that

$$\varphi^t(e) - \lambda_s\langle\varphi_s^t\rangle \in V + B_{\delta/2} \qquad \text{for every } \varphi^t \in \Phi \text{ with } t \geqslant t_1,$$

where e is the identity of G. In fact, for the case (1),

$$\begin{aligned}\varphi^t(e) - \lambda_s\langle\varphi_s^t\rangle &= x_t - \sum_{i=1}^{n} T_{s_i}x_t \\ &= \sum_{i=1}^{n} a_i(x_t - T_{s_i}x_t) \in V\end{aligned}$$

for every $t \geqslant t_1 = t_0$. For the case (2), take $t_1 \geqslant t_0$ such that

$$\|T_{s_i}x_t - x_{s_i+t}\| < \delta/2$$

for every $t \geqslant t_1$ and $i = 1, 2, \ldots, n$. Then,

$$\begin{aligned}\varphi^t(e) - \lambda_s\langle\varphi_s^t\rangle &= \sum_{i=1}^{n} a_i(x_t - T_{s_i}x_t) + \sum_{i=1}^{n} a_i(T_{s_i}x_t - x_{s_i+t}) \\ &\in V + B_{\delta/2}\end{aligned}$$

for every $t \geqslant t_1$. Therefore

$$\begin{aligned}&\sup\{\langle w+v, y_t^*\rangle + \delta : w \in W', v \in V\} \\ &\quad < |\langle x_{k_t}, y_t^*\rangle| \\ &\quad \leqslant |\langle\varphi^{k_t}(e) - \lambda_s\langle\varphi_s^{k_t}\rangle, y_t^*\rangle| + |\langle\lambda_s\langle\varphi_s^{k_t}\rangle \\ &\qquad - (\mu_{\alpha_0})_s\langle\varphi_s^{k_t}\rangle, y_t^*\rangle| + |\langle(\mu_{\alpha_0})_s\langle\varphi_s^{k_t}\rangle, y_t^*\rangle| \\ &\quad \leqslant \sup_{v \in V}\langle v, y_t^*\rangle + \delta/2 + \left\|\int \varphi_s^{k_t}\, d\lambda(s) - \int \varphi_s^{k_t}\, d\mu_{\alpha_0}(s)\right\| \cdot \|y_t^*\| \\ &\qquad + \sup_{w \in W'}\langle w, y_t^*\rangle \\ &\quad \leqslant \sup_{v \in V}\langle v, y_t^*\rangle + \delta + \sup_{w \in W'}\langle w, y_t^*\rangle\end{aligned}$$

for any $t \geqslant t_1$. This is a contradiction.

COROLLARY 5. *Let E, C, and S be as in Corollary 1. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in C such that $x_n - Sx_n$ converges, as $n \to \infty$, weakly to 0. Then, for each weak neighborhood W of $F(S)$, $x_n \in W$ for sufficiently large n.*

COROLLARY 6. *Let E, C, and $\{S(t): t \geqslant 0\}$ be as in Corollary 2. Let $\{x_t: t \geqslant 0\}$ be a net in C such that $t \to x_t$ is uniformly continuous,*

$\lim_{t\to\infty} \sup_{0 \leqslant u \leqslant 1} \|S(u)\,x_t - x_{t+u}\| = 0$, *and* $x_t - S(s)\,x_t$ (*or equivalently* $x_t - x_{t+s}$) *converges, as* $t \to \infty$, *weakly to* 0 *for each* $s \geqslant 0$. *Then, for each weak neighborhood* W *of the set of common fixed points of* $\{S(t): t \geqslant 0\}$, $x_t \in W$ *for sufficiently large* t.

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